Letters

Comments on "Conformal Transformations Combined with Numerical Techniques, with Applications to Coupled-Bar Problems"

P. A. A. LAURA

Abstract—**The writer lists additional references on recent developments on the conformal mapping method which may be of interest to microwave specialists.**

The writer heartily congratulates Levy for his very interesting and important contribution[1] which also deserves attention from specialists in other fields of technology and applied science.

It is also the purpose of the present paper to point out the existence of recently published technical literature where other conformal mapping applications and approaches have been described. It is felt that these conformal mapping techniques may be also of interest to microwave engineers in some instances.

As stated by Levy, classical applications of conformal mapping deal with solutions of problems of mathematical physics governed by Laplace's equation. The fundamental advantage of the methodology lies in the fact that the equation remains invariant under transformation.

Late in the Nineteenth century the great French mathematician E. Goursat showed that the solution of the biharmonic equation:

$$\nabla^4\phi=0 \tag{1}$$

is expressible in the form

$$\phi=\mathrm{Re}[\bar{z}\psi(z)+x(z)] \tag{2}$$

where Re means "real part of," $\bar{z}$ denotes $x-iy$ and $\psi(z)$ and $x(z)$ are appropriate analytic functions [2]. These approach was later on used by Muschelisvili [3] when constructing a now classical methodology for solving plane stress analysis problems in the case of domains of complicated boundary shape based on the method of conformal transformation.

It is quite interesting to point out that the first analytical studies on stresses and strains in solid propellant rocket grains early in the sixties were made using the conformal mapping approach [4].

In the last two decades numerous nonclassical applications of the method of conformal mapping have been developed for a wide family of problems not governed by Laplace's equation:

theory of elasticity (isotropic and anisotropic media);
determination of cutoff frequencies of electromagnetic waveguides (also applications to mathematically related fields such as acoustical waveguide theory and membrane vibrations);
radio propagation in the atmosphera;
flow and heat transfer in ducts of arbitrary shape;
unsteady heat conduction problems (also diffusional processes following Fick's law);
supersonic flows;
solidification problems;
plate theory (flexure, stability and vibrations problems);
ion optics;
propagation of optical modes in dielectric fibers;
diffraction of electromagnetic waves;
atomic physics (theory of reactive collisions); etc.

An informative review of these applications can be found in [5].

Finding the analytic function which performs the desired mapping of the given simply or doubly connected domain onto a simpler shape in another plane is obviously one of the most important steps in these applications.

In the case of simply connected domains Wilson [4] determined mapping functions by solving an integral equation of the Fredholm type. For doubly connected domains, a coupled system of two integral equations must be solved [6], [7].

Ives has developed several important conformal mapping concepts [8]. His most significant result is the introduction of a new class of transformation, of which the von Karman–Trefftz transformation is a special case. Ive's paper deals with the numerical solution of the transonic flow equations in two dimensions.

A significant accomplishment in the development of conformal mapping techniques is due to Halsey [9].

In order to calculate the two-dimensional flow about multielement airfoils he develops a conformal mapping technique based on the use of fast Fourier Transforms. Unlike other mapping methods, very arbitrary multielement airfoils (with any number of elements) can be analyzed. The airfoil components are transformed to the same number of disjoint circles. The flow is determined in the multiple-circle plane and then transformed back to the physical plane.

One may conclude that from Ptolomy's conformal transformation of the celestial sphere into a plane developed almost 2000 years ago [10] to sophisticated determinations of mapping functions and their application to complex scientific and technological problems in a wide variety of fields, Man has accomplished significantly in this respect in the last twenty centuries.

Reply by[2] *R. Levy*[3]

Dr. Laura has drawn attention to certain interesting applications of conformal transformation which were not covered in my paper[1]. Readers may wish to pursue his references, some of which are unfortunately rather obscure. Dr. Laura's own paper [10] gives a number of more accessible references, some of which are listed below [11]–[15]. It appears that these are all concerned with problems which are somewhat loosely related to that considered in [1]. The latter is restricted to two dimensional solutions of Laplace's equation. It points out that conformal transformations may be applied to eliminate field singularities and, in contradiction to text book statements, need not in fact be restricted to

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[1]R. Levy, *IEEE Trans. Microwave Theory Tech*, vol. MTT-28, pp. 369–375, Apr. 1980.

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shapes which are transformable into perfectly regular boundaries. The object is to transform to boundaries which may be nonregular in the mathematical sense (i.e., no exact closed form solution exists), but which are smooth and slowly varying, so that elementary numerical solutions may be applied to give highly accurate results. Since singularities and sharp corners have been eliminated, good results are obtained from numerical techniques using relatively coarse meshes. The method should be used in conjunction with the network analog theory described in [9], which shows how to avoid slowly convergent and uncertain iterative numerical techniques.

The references cited by Laura appear to be concerned mainly with the solution of more difficult field theory problems, e.g., determination of the cut off frequencies of waveguides of unusual cross section [12]–[16]. As Dr. Laura states, Laplace's equation is invariant under a conformal transformation, but this is not the case for more general wave equations. Here conformal transformations may be applied to transform the boundaries into regular shapes, but the field equations become far more complex. In a sense one is transforming one type of complex problem into another, transferring the difficulty from the boundary to the form of equation. A good example of this is described in [14], where the groove guide is transformed into a parallel-plate guide filled with a nonuniform anistropic medium. In solving these transformed problems, complicated Fourier or integral equation techniques need to be employed. The methods have proven quite successful in many instances, but may be considered to be rather specialized.

I would like to take this opportunity to make a correction to eq. (24) of [1] which should read

$$C_g + 2C_f = C_I. \tag{24}$$

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Comments on "The Matched Feedback Amplifier: Ultrawide-Band Microwave Amplification with GaAs MESFET's"

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In the above paper [1], the bandwidth of a single GaAs MESFET was extended by means of series inductances (L_D and L_{FB}) in the feedback path.

The authors used voltage vector diagrams to illustrate the action of this feedback and that of a simpler circuit in which L_D and L_{FB} of Fig. 1(a) were both set to zero, leaving only R_{FB} between drain and gate. A polar plot of the feedback current for $L_D=0.6$ nH and $L_{FB}=0.45$ nH [c.f., 1, fig. 2 curve D] was also shown [1, fig. 4].

The behavior of their circuit diagram [1, fig. 1(a)], (reproduced here as Fig. 1), can perhaps be better understood if conventional loop gain plots, rather than their vector diagrams, are used. Loop gain is a well-known concept in feedback amplifiers, and can be expected to be familiar to readers of the paper [1]. The appropriate point at which to break the loop is in the branch which contains R_{FB}. If the loop gain was to be measured by means of a 50-Ω network analyzer, R_{FB} could be replaced by two resistors of some 80-Ω connected to the input and output of the analyzer, to obtain the best accuracy in measuring the quantities S_{11} and S_{22}. This was also done for the present analysis, and the resulting open-loop gains computed, corresponding to the authors' curves A and D [1, fig. 2]. These loop gains are shown in Fig. 2. If we examine Fig. 2 it is clear that the above values of L_D and L_{FB} have kept the magnitude of the loop gain more nearly constant. The phase of this loop gain goes through 0° just above 15 GHz (compared with 13.75 GHz cited), at which point the "feedback" is purely positive, but less than -15 dB. In common parlance, the gain margin is some 15 dB, and the gain margin frequency around 15 GHz. A phase margin cannot be defined since the magnitude of the loop gain never exceeds one in the frequency range shown. Comparing Fig. 2 with [1, fig. 4] it is evident that the phase angles of the loop gain and of the feedback current differ by some 180°. Finally, because of the low values of the loop gain (LG), the quantity $|1-LG|$ is close to unity, and there can be little gain enhancement due to "positive feedback."

An improved method of assessing feedback, especially in the frequency range covered by this amplifier has recently been described [2]–[5]. It is known as the "embedding network" method, and is mainly intended for multiloop amplifiers characterized from s-parameter measurements, which yield the best, commercially obtainable accuracy. A circuit diagram can, however, be used with some loss of realism arising from the artificiality of most circuit diagrams at such frequencies.

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